

# Refinement Strategies and Approximation Errors for Tetrahedral Elements

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**Abstract** — Two tetrahedral mesh refinement algorithms proposed by Zhang are compared using the new singular value criterion for the element edge shape matrix. The ‘short-edge subdivision’ scheme is better than ‘labeled edge subdivision’. Validity of the singular value criterion is confirmed and its geometric implications are investigated.

**Index terms**—tetrahedral elements, refinement, accuracy estimate, approximation, interpolation, singular value criterion

## I. INTRODUCTION

This paper completes a sequence of studies [1-3] of the influence of finite element shape on the interpolation accuracy. The general maximum eigenvalue condition [1] was applied to tetrahedral elements and ultimately led to a precise characterization of the tetrahedral shape by the minimum singular value of the ‘edge shape matrix’ [2].

Two separate but related issues are now addressed. First, the minimum singular value condition is used to evaluate two refinement strategies proposed by Zhang [4]. This comparative evaluation also illustrates the accuracy and usefulness of the minimum singular value condition. Second, geometric implications of the minimum singular value condition are further investigated extending the analysis of [2].

## II. TETRAHEDRAL MESH REFINEMENT

Adaptive mesh refinement is widely used in modern finite element algorithms to achieve higher accuracy of the numerical solution. While subdivision of 2D elements is simple, the 3D case, in particular for tetrahedral elements, is not at all trivial; careless subdivision may lead to degenerate elements. S. Zhang [4] proposed two schemes – ‘labeled edge subdivision’ and ‘short-edge subdivision’ – guaranteeing that tetrahedral elements do not degenerate in the refinement process.

The initial stage of both methods is the same: the edge midpoints are connected, producing four corner tetrahedra and a central octahedron (Fig. 1, see also [4]). The octahedron can be further subdivided into four tetrahedra in three different ways [4], namely, by adding one of the edges  $aa'$ ,  $bb'$  or  $cc'$ .

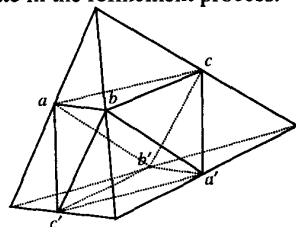


Fig. 1. Subdivision of a tetrahedral element into four corner tetrahedra and the central octahedron.

The difference between Zhang’s two refinement schemes

is in the way this additional edge is chosen. The ‘labeled edge subdivision’ algorithm employs a direct numbering scheme for nodes being generated and subdivides the octahedron in accordance with this numbering (see [4] for details). In the ‘short edge subdivision’ algorithm the *shortest* of the three possible interior edges  $aa'$ ,  $bb'$ ,  $cc'$  is selected.

## III. ELEMENT SHAPE AND ERROR ESTIMATES

S. Zhang points out that “in general, it is not simple to find the measure of degeneracy for a given tetrahedron” [4] and uses as such a measure the ratio of the maximum edge length to the radius of the inscribed sphere.

In [2], a more precise characterization of tetrahedral shape was given. Consider the  $3 \times 6$  ‘edge shape matrix’  $E$  whose columns are the unit vectors  $e_i$  ( $1 \leq i \leq 6$ ) along the edges of a tetrahedral element (in either of the two possible directions):

$$E = [e_1 | e_2 | e_3 | e_4 | e_5 | e_6] = \begin{bmatrix} e_{1x} & e_{2x} & e_{3x} & e_{4x} & e_{5x} & e_{6x} \\ e_{1y} & e_{2y} & e_{3y} & e_{4y} & e_{5y} & e_{6y} \\ e_{1z} & e_{2z} & e_{3z} & e_{4z} & e_{5z} & e_{6z} \end{bmatrix} \quad (1)$$

The minimum singular value of this matrix,  $\sigma_{\min}(E) \geq 0$ , characterizes the ‘level of degeneracy’ of the element, or the ‘level of linear independence’ of the six unit edge vectors. In particular, this singular value is zero only when *all six* vectors  $e_i$  are coplanar (i.e. the tetrahedron is completely degenerate).

Alternatively,  $\sigma_{\min}(E)$  can be viewed as a ‘condition number’ of the transformation from the Cartesian components  $H_{xyz} \in R^3$  of a vector  $H$  to its projections  $H_e \in R^k$  onto  $k$  unit edge vectors (only  $k=3$  and  $k=6$  will actually be needed)<sup>1</sup>. The two representations of  $H$  are related as

$$H_e = E^T H_{xyz} \quad (2)$$

for  $k=6$ , and in a similar way for other values of  $k$  (with the appropriate  $3 \times k$  submatrix of  $E$  in (2)).

Using (2) and the fact that the singular values of  $E$  are equal to the square roots of the eigenvalues of  $EE^T$  (or of the nonzero eigenvalues of  $E^T E$ , for that matter) [5], one may characterize  $\sigma_{\min}(E)$  as [2]

$$\sigma_{\min}(E) = \min_{\|H_{xyz}\|_{E^3}=1} \|H_e\|_{E^6} \quad (3)$$

where the minimum is reached when  $H_{xyz}$  is the minimum eigenvector<sup>2</sup> of  $EE^T$ .

The finite element interpolation error  $\epsilon_l$  over the element is governed by the same singular value  $\sigma_{\min}(E)$  [2]:

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<sup>1</sup> We do not exclude degenerate cases where the  $k$  unit vectors are linearly dependent.

<sup>2</sup> I.e. the eigenvector corresponding to the minimum eigenvalue.

$$\varepsilon_I \leq ch \sigma_{\min}^{-1}(E) \sqrt{V} \quad (4)$$

where  $h$  is the element diameter (i.e. the maximum edge),  $V$  is the volume of the element, and the interpolation error is measured in the energy seminorm

$$\varepsilon_I = \left[ \int_{\text{element}} |\nabla u_I - \nabla u_*|^2 dV \right]^{1/2} \quad (5)$$

( $u_*$  is the exact scalar potential being interpolated and  $u_I$  is the standard first order nodal interpolant). The proof [2] of (4) is based on the maximum eigenvalue criterion of [1].

A priori error estimate (4) can be interpreted as a convergence condition for the interpolation of the scalar potential on first order tetrahedral elements (or, equivalently, of conservative fields on Whitney edge elements [6]). If on a family of meshes with the maximum mesh size tending to zero the minimum singular value of all element edge shape matrices is bounded away from zero, then the convergence of the FE interpolation is assured.

#### IV. COMPARISON OF ACCURACY CRITERIA

##### A. General Considerations

From the mathematical point of view, for a sequence of tetrahedral meshes with the maximum mesh size tending to zero, one looks for sufficient conditions for the interpolation error to tend to zero. If  $X$  and  $Y$  are two such conditions and  $X$  follows from  $Y$  (i.e. whenever  $Y$  is satisfied,  $X$  is guaranteed to be satisfied as well), then  $X$  is a *weaker* (less restrictive) sufficient condition than  $Y$ .

Generally speaking, one seeks the *least restrictive*, i.e. the weakest, sufficient condition(s). For example, the condition that the element be regular (equilateral) is valid and sufficient, but way too restrictive and therefore impractical.

Apart from purely academic interest, the least restrictive sufficient condition for convergence of FE interpolation would establish a practical 'benchmark' for tetrahedral shape. We argue that the minimum singular value of the edge shape matrix is such a benchmark. Dihedral angles or angles between edges are superfluous. It is the level of linear independence of six unit edge vectors, characterized by the minimum singular value, that governs the interpolation error.

What follows is a comparison of the singular value criterion with various other geometric measures. Since we consider measures that depend only on the element shape, not size, the tetrahedral elements are always assumed to be scaled to the unit maximum edge.

##### B. The Minimum Singular Value vs. Angle Conditions

Suppose that a (scaled) tetrahedral element  $T$  undergoes a continuous transformation, parametrized by a parameter  $t$ , from its initial state at  $t = 0$  to a certain degenerate state with the zero minimum singular value at  $t = 1$ . For any sequence  $t_n \rightarrow 1$  the singular value condition is thus violated. The fol-

lowing statement shows that the singular value criterion is equally or less stringent than some conceivable angle conditions:

if  $E_n$  is the edge shape matrix of the tetrahedron  $T_n \equiv T(t_n)$  and  $\sigma_{\min}(E_n) \rightarrow 0$  as  $n \rightarrow \infty$ , then

- (a) all solid (trihedral) angles of  $T_n$  tend to either zero or  $2\pi$ ;
- (b) all angles between edges and faces, with a possible exception of one face, tend to either zero or  $\pi$ .

The proof is based on the following observation of [2]. Without loss of generality, any tetrahedral element can be translated to have one of its nodes (node 1) at the origin of a Cartesian system, scaled to make its longest edge (1-2) equal to one, and rotated to have the minimum eigenvector of  $EE^T$  run along the  $z$  axis. Then, as shown in [2], nodes 2-4 of the tetrahedron will be 'sandwiched' between the planes  $z = \pm\sigma_{\min}(E)$  and, due to scaling, will be contained within the unit ball centered at the origin.

For  $n \rightarrow \infty$ , projections of the four nodes onto the  $xy$ -plane converge to a quadrilateral figure  $1'2'3'4'$  (Fig. 2), node 1 being at the origin and projection  $2'$  lying on the unit circle due to scaling.

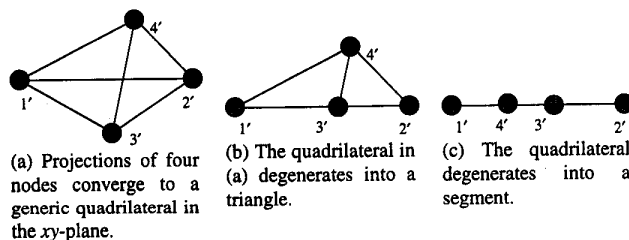


Fig. 2. Projections of the tetrahedral nodes on the  $xy$ -plane.

In the generic case (Fig. 2a), when the quadrilateral is not degenerate, all faces lie in the  $xy$ -plane (in the limit) and all solid angles and all angles between edges and faces tend to zero,  $2\pi$  or  $\pi$ . If projections of three nodes ( $1'2'3'$ , Fig. 2b) lie on one straight line, all solid angles are zero or  $2\pi$ , and all angles between edges and faces approach zero or  $\pi$ , except possibly for face 123. Finally, if all projections  $1'2'3'4'$  lie on one line, statements a) and b) hold for all angles, even if some of the projections happen to coincide (say,  $4' \equiv 1'$ ).

*Remark.* The underlying continuous parameter  $t$  was in fact redundant. For any infinite sequence of scaled tetrahedral elements, one can select a subsequence with the  $xy$ -projections of their nodes converging to a quadrilateral in the  $xy$ -plane (due to compactness of the unit circle).

Any geometric condition equivalent to the minimum singular value criterion must ensure that whenever the condition is violated for a sequence of elements, all tetrahedral edges in the limit become coplanar. One valid condition is for the maximum solid angle to be bounded away from  $2\pi$ . Indeed, if this condition were violated, the three edges forming the

large trihedral angle would have to lie, in the limit, in one plane, and hence all six edges would approach that plane, too.

Another valid criterion is given by Křížek [7]: the maximum dihedral angle *and* the maximum planar angle in all four triangular faces should be bounded away from  $\pi$ . It can be shown [8] that Křížek's criterion is asymptotically equivalent to the minimum singular value condition.

### C. The Minimum Singular Value vs. the Trihedral Volume

First, consider three unit edge vectors corresponding to a common tetrahedral node. There is a  $3 \times 3$  submatrix  $E_3$  of  $E$  associated in the obvious way with these three edges. The volume of a parallelepiped based on the three unit vectors is

$$V_3 = \det(E_3) \quad (6)$$

Both  $\sigma_{\min}(E_3)$  and  $V_3$  characterize the level of linear dependence of the three unit vectors, suggesting a connection between these two measures. Recalling that the product of the eigenvalues is equal to the determinant, and the sum of the eigenvalues is equal to the trace, one has

$$\begin{aligned} [\sigma_1(E_3) \sigma_2(E_3) \sigma_3(E_3)]^2 &= \lambda_1(E_3^T E_3) \lambda_2(E_3^T E_3) \lambda_3(E_3^T E_3) \\ &= \det(E_3^T E_3) = \det^2(E_3) = V_3^2, \quad \text{i.e.} \\ \sigma_1(E_3) \sigma_2(E_3) \sigma_3(E_3) &= V_3 \end{aligned} \quad (7)$$

Similarly,

$$\begin{aligned} \sigma_1^2(E_3) + \sigma_2^2(E_3) + \sigma_3^2(E_3) &= \lambda_1(E_3^T E_3) + \lambda_2(E_3^T E_3) \\ &+ \lambda_3(E_3^T E_3) = \text{tr}(E_3^T E_3) = 1+1+1 = 3 \end{aligned} \quad (8)$$

From (8), one immediately obtains

$$1 \leq \sigma_{\max}^2(E_3) \leq 3$$

and therefore, assuming  $\sigma_{\max} = \sigma_1 \geq \sigma_2 \geq \sigma_3 = \sigma_{\min}$ ,

$$\begin{aligned} \sigma_{\min}^4(E_3) &\leq \sigma_{\min}^2(E_3) \sigma_2^2(E_3) \leq \sigma_1^2(E_3) \sigma_2^2(E_3) \sigma_3^2(E_3) \\ &= V_3^2 \leq \sigma_1^2(E_3) \sigma_1^2(E_3) \sigma_3^2(E_3) \leq 9 \sigma_{\min}^2(E_3) \\ \Rightarrow V_3/3 &\leq \sigma_{\min}(E_3) \leq \sqrt{V_3} \end{aligned} \quad (9)$$

The right inequality indicates that  $\sigma_{\min}$  and  $V_3$  could be of different 'orders of magnitude'. Indeed, if, for example,

$$E_3 = \begin{pmatrix} 1 & \cos \varepsilon & \cos^2 \varepsilon \\ 0 & \sin \varepsilon & \cos \varepsilon \sin \varepsilon \\ 0 & 0 & \sin \varepsilon \end{pmatrix} \quad (10)$$

then for  $\varepsilon \rightarrow 0$ ,  $V_3 = \det(E_3) = \sin^2 \varepsilon = O(\varepsilon^2)$ , and it can be verified that  $\sigma_{\min}(E_3) = O(\varepsilon) = O(V_3^{1/2})$ .

On the other hand, for the matrix

$$E_3 = \begin{pmatrix} 1 & 1/2 & -1/2 \cos \varepsilon \\ 0 & \sqrt{3}/2 & \sqrt{3}/2 \cos \varepsilon \\ 0 & 0 & \sin \varepsilon \end{pmatrix} \quad (11)$$

$\sigma_{\min}(E_3)$  and  $V_3$  are of the same asymptotic order  $O(\varepsilon)$ , consistent with the left inequality in (9).

It is easy to show that  $\sigma_{\min}(E_3)$  in (9) is related to  $\sigma_{\min}(E)$  as

$$\max \sigma_{\min}(E_3) \leq \sigma_{\min}(E)$$

where the maximum is taken over all  $3 \times 3$  submatrices  $E_3$  of  $E$ .

Due to a 'nonlinear' relationship (9) between  $V_3$  and  $\sigma_{\min}$ ,  $V_3$  is expected to be a less accurate *a priori* shape measure than  $\sigma_{\min}$ .

## V. NUMERICAL EXPERIMENTS

The experiments were designed to compare Zhang's refinement algorithms for tetrahedra and to verify the accuracy of the minimum singular value condition.

### A. Refinement of Individual Elements

In the first set of experiments, refinement of an individual tetrahedron is considered. First, several levels of labeled edge subdivision are applied and then refinement is repeated with short edge subdivision. Statistics for  $\sigma_{\min}(E)$  for all elements appearing in the process of refinement are collected.

S. Zhang showed [4] that for tetrahedra without obtuse planar angles between edges both refinement schemes are equivalent, provided the initial refinement is the same (i.e. for a certain numbering of nodes of the initial element).

Table I summarizes the results for the initial tetrahedron  $\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$  with three mutually orthogonal edges. (Other shapes do appear in the process of refinement.) The minimum, maximum and average values of the error criterion  $\sigma_{\min}^{-1}(E_i)$  over all elements  $i$  on each refinement level are shown.

TABLE I. SHAPE CHARACTERISTICS OF ELEMENTS:  
REFINING A TETRAHEDRON WITH THREE ORTHOGONAL EDGES

	Refine- ment level	Minimum of $\sigma_{\min}^{-1}$	Average of $\sigma_{\min}^{-1}$	Maxi- mum of $\sigma_{\min}^{-1}$
	0	1.000	1.000	1.000
Labeled	1	0.967	1.095	1.414
edge	2	0.967	1.119	1.414
refinement	3	0.967	1.125	1.414
	4	0.967	1.127	1.414
Short edge refinement	Same as for labeled edge refinement above			

<sup>3</sup> The MACSYMA symbolic algebra package was used for verification. MACSYMA® is a registered trademark of Macsyma Inc.

Table II refers to the tetrahedron  $\{(0,0,0), (0,0,1), (1,0,0), (-1,1,0)\}$  with an obtuse angle between edges. The ordering of nodes is such that the initial labeled edge and short edge refinements coincide. However, the subsequent refinement levels are different, with the short edge refinement performing better. (Results of the next section confirm this conclusion.) While the *maximum* values of  $\sigma_{\min}^{-1}$  for both subdivision algorithms are equal, i.e. the 'worst' elements have the same shapes, the *average* criterion  $\sigma_{\min}^{-1}$  for the short edge scheme is smaller and even decreases with the level of refinement.

TABLE II. SHAPE CHARACTERISTICS OF ELEMENTS: REFINING A TETRAHEDRON WITH AN OBTUSE ANGLE

	Refinement level	Minimum of $\sigma_{\min}^{-1}$	Average of $\sigma_{\min}^{-1}$	Maximum of $\sigma_{\min}^{-1}$
Labeled edge refinement	0	1.647	1.647	1.647
	1	1.414	1.680	2.013
	2	1.414	1.689	2.013
	3	1.414	1.691	2.013
	4	1.414	1.691	2.013
Short edge refinement	0	1.647	1.647	1.647
	1	1.414	1.680	2.013
	2	0.967	1.583	2.013
	3	0.967	1.479	2.013
	4	0.967	1.393	2.013

### B. A Model Boundary Value Problem

In the second set of experiments, boundary value problems with known theoretical solutions were solved using the multigrid method described in [9], with global mesh refinement by either 'labeled edge' or 'short edge' subdivision.

As a test problem, we considered the Laplace equation in a cube, with inhomogeneous Dirichlet boundary condition on the whole surface, and a known theoretical solution  $u_*$  of the form  $\sin ax \cos by \cosh((a^2+b^2)^{1/2} z)$  or similar [3]. The error was evaluated in the seminorm (5) corresponding to the energy of the field. A typical set of results is presented in Fig. 3 in a somewhat nonstandard way. Actual *global* errors (labeled as  $\blacktriangle$  and  $\bullet$  for two different problems) and their singular value estimates<sup>4</sup> ( $\blacklozenge$ ) are plotted. The values for the short edge subdivision scheme are shown on the vertical axis and the respective values for labeled edge subdivision on the horizontal axis. Thus in the lower 45° half of the first quadrant the short edge subdivision is better, whereas in the upper half the labeled edge subdivision would be preferable.

Each data point in Fig. 3 corresponds to a particular mesh. All data were scaled to make the values for the initial coarse mesh equal to one. As the mesh is refined, global errors generally decrease and the data points approach the origin.

It is clear from Fig. 3 that short edge subdivision was better than labeled edge subdivision in all cases<sup>5</sup>. It is also evident that the minimum singular value criterion has given qualitatively accurate predictions.

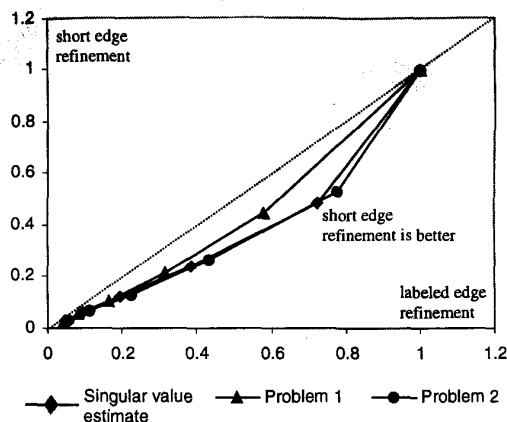


Fig. 3. Actual errors and the minimum singular value for short edge vs. labeled edge refinement. Problem 1:  $u_* = \sin(\pi x/2) \cos(\pi y/2) \cosh(\sqrt{2} \pi z/2)$  Problem 2:  $u_* = \sin(\pi y/2) \cos(\pi x/2) \cosh(\sqrt{2} \pi z/2)$

## IV. CONCLUSIONS

Comparison of the two refinement schemes using the minimum singular value criterion shows that short-edge subdivision is better than labeled edge subdivision. The minimum singular value of the edge shape matrix is closely related to several geometric characteristics and may be viewed as a benchmark *a priori* measure of tetrahedral shape.

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<sup>4</sup> The singular value estimates were calculated as root mean square averages of element-wise estimates (4).

<sup>5</sup> The initial node numbering for labeled edge subdivision was chosen at random to avoid potential biases. Results for the two refinement schemes are therefore different.