Symbolic Algebra as a Tool for Understanding Edge Elements

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***Abstract*—The paper describes a methodology for systematic comparison of tetrahedral edge element spaces using symbolic algebra. Various known edge element families, with their respective spaces and irrotational subspaces, are studied. In addition, symbolic algebra facilitates analysis of some novel hexahedral elements and of a new "prolongation of local gradients" condition for spectral convergence (the absence of "spurious modes").**

***Index Terms*—Edge elements, spectral convergence, spurious modes, symbolic algebra.**

I. Introduction

TWO separate but related topics are considered. First, a methodology for systematic comparison of tetrahedral edge element spaces using symbolic algebra is introduced in Section II. In 1980 and 1986, Nedelec proposed two families of tetrahedral and hexahedral edge elements [1], [2] without explicitly specifying a basis, and later on quite a number of possible bases were given by different researchers (e.g., [3]–[10]). The existing comparative analysis of these options [4], [6] is very helpful, but it is still not easy to find out from the literature whether the Nedelec vector spaces and their curl's are the same as, say, the Ahagon spaces or Webb or Lee or Ren or Kameari or Yioultsis spaces.

The second topic is spectral convergence, i.e., the absence of "spurious modes" [11]–[13]. In Section V, symbolic algebra helps to analyze a new condition for spectral convergence, called prolongation of local gradients (PLG) [13], in the curious case of crisscross meshes.

The paper is limited in scope. Several very interesting subjects, such as *hp*-refinement with Demkowicz–Vardapetyan elements (see [14] and references therein) and Hiptmair's new general perspective on high-order edge elements [15], could not be included. The interested reader is also referred to the accompanying papers [13], [18].

II. General Methodology

Consider a set of vector functions $\{u_i\}$ $(1 \leq i \leq n)$ in a finite element—for definiteness, a tetrahedron T—spanning an edge-element space U. Let $\{w_j\}$ $(1 \leq j \leq m \leq n)$ span another edge-element space W that may or may not be different from U.

Remark 1: It will be convenient *not* to assume that $\{u_i\}$ or $\{w_j\}$ are necessarily linearly independent.

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Notational Convention. Euclidean coefficient vectors are underscored to distinguish them from the respective vector functions; parentheses denote the Euclidean inner product for Euclidean vectors and the $L_2(T)^3$-product for vector functions.

Let the dimension of U be $\dim U \equiv d_U \leq n$ and $\dim W \equiv d_W \leq m$. The usual Gram matrices M^{uu} and M^{ww} (also known as "mass matrices" in finite-element methods) have the entries $M_{ij}^{uu} = (u_i, u_j)$ and $M_{ij}^{ww} = (w_i, w_j)$. By definition of the Gram matrix, for any pair of coefficient vectors $\underline{p} \in R^n$, $\underline{q} \in R^n$ one has

$$(M^{uu}\underline{p}, \underline{q}) = (p, q) \text{ and in particular } (M^{uu}\underline{p}, \underline{p}) = (p, p) \quad (1)$$

where p and q are linear combinations of functions in their respective sets: $p = \sum_{i=1}^{n} p_i^u u_i$, $q = \sum_{i=1}^{n} q_i^u u_i$. An analogous relationship of course holds for M^{ww}.

It follows immediately from (1) that any coefficient vector $\underline{p}$ in the nullspace of the Gram matrix (and only these vectors) correspond to the zero vector function in the respective finite-element (FE) space. The nullspace of M^{uu} has dimension $n_0 = n - d_U$, and the nullspace of M^{uu}-dimension $m_0 = m - d_W$.

Further, we define the *mutual mass matrix* of $\{u_i\}$ and $\{w_j\}$ as the (2×2 block) Gram matrix of all $n+m$ functions $\{u_i, w_j\}$

$$M = \begin{pmatrix} M^{uu} & -M^{uw} \\ -M^{wu} & M^{ww} \end{pmatrix}$$

where $M_{ij}^{uw} = (u_i, w_j)$, $M^{wu} = (M^{uw})^{\mathrm{T}}$, and the minus signs are included for convenience in later analysis.

For any two coefficient vectors $\underline{p} \in R^{n+m}$, $\underline{q} \in R^{n+m}$ partitioned in a natural way as $\underline{p} = [\underline{p}^u, -\underline{p}^w]$, $\underline{q} = [\underline{q}^u, -\underline{q}^w]$ one has

$$(M\underline{p}, \underline{q}) = (p, q) \text{ and in particular } (M\underline{p}, \underline{p}) = (p, p) \quad (2)$$

where p and q are now linear combinations of both sets of functions

$$p = \sum_{i=1}^{n} p_i^u u_i - \sum_{j=1}^{m} p_j^w w_j \quad q = \sum_{i=1}^{n} q_i^u u_i - \sum_{j=1}^{m} q_j^w w_j.$$

As before, the nullspace of the mutual mass matrix corresponds to the zero vector function in the FE space. The null vectors can be subdivided into three categories: 1) n_0 linearly independent vectors $\underline{p} = [\underline{p}^u, 0]$, $\underline{p}^u$ being in the nullspace of M^{uu}; 2) m_0 linearly independent vectors $\underline{p} = [0, -\underline{p}^w]$; 3) $d_{U \cap W} \equiv \dim(U \cap W)$ linearly independent vectors $\underline{p} = [\underline{p}^u \neq 0, -\underline{p}^w \neq 0]$. Part 3) is particularly important, as it corresponds to vectors that can be represented as a linear combination of either $\{u_i\}$ or, separately, $\{w_j\}$.

The null vectors of M can be arranged as columns of the $(n+m) \times (n_0 + m_0 + d_{U \cap W})$ matrix N (so that $MN = 0$) that can be partitioned in a natural way as a 2×3 block matrix (Table I). Columns of N_c^u, in particular, represent $U \cap W$ in terms of the $\{u_i\}$ set

$$\text{Range}\,(N_c^u) \leftrightarrow U \cap W \text{ in terms of } \{u_i\}. \quad (3)$$

One way to generate a subspace of U that is *not* in W is to compute vectors orthogonal to the columns of N_c^u—and such vectors form the nullspace of $(N_c^u)^T$

$$(\text{Range}\,(N_c^u))^{\perp} = \text{Null}\,(N_c^u)^T \leftrightarrow \text{a subspace of } U \not\subset W. \quad (4)$$

Alternatively, one can construct the L_2-orthogonal complement of $U \cap W$ in U. Noting that the mass matrices relate Euclidean and L_2 inner products as in (1) and (2), one obtains

$$\text{Null}\left[(N_c^u)^T M^{uu}\right] \leftrightarrow (U \cap W)_{L2}{}^{\perp} \text{ in terms of } \{u_i\}. \quad (5)$$

Obviously, similar relationships hold if the roles of U and W are interchanged.

In most electromagnetic problems, approximation of the curl is as important as that of the field itself. To this end, we consider the Gram matrices of $\nabla \times u_i$ and $\nabla \times w_j$ that we shall call the (mutual) curl-matrices. The nullspace of a curl-matrix corresponds to irrotational fields in the respective FE space. The *curl*-spaces spanned by $\{\nabla \times u_i\}$ and $\{\nabla \times w_j\}$ in general have strictly lower dimensions than the respective full edge-element spaces, which explains the need for Remark 1.

Most properties of interest to us are invariant with respect to affine transformations of coordinates (see [12] for rigorous definitions and proofs) and can, therefore, be studied for the reference tetrahedron with three orthogonal unit edges. For properties that do depend on shape (such as condition numbers of Gram matrices and interpolation errors), it still makes sense to treat this dependence as a separate issue.

Obviously, theoretical analysis must rely on an *exact*, not numerical, representation of the Gram matrix nullspaces for the reference tetrahedron. It has come as a pleasant surprise that this task for second order elements can be handled very effectively by symbolic algebra,[1] with the analysis time for any pair of edge-element spaces on the order of minutes at most. Specific families of elements are considered below.

III. Comparison of Various Edge-Element Families

Most of the results reported below are already known but scattered through the literature. Symbolic algebra, however, provides a unified perspective and a simple way of comparative analysis of edge element spaces. In particular, all pairwise mutual mass matrices of the Ahagon, Ren, and Yioultsis [2] bases have the nullspace of dimension 20, and consequently their rank is also $20\ (= 40 - 20)$, i.e., the union of, say, Ahagon $\cup$ Ren spaces has the same dimension (20) as the Ahagon space alone.

TABLE I
Matrix of Null Vectors of M, Partitioned

	n_0	m_0	$d_{U \cap W}$
n	N_a^u	0	N_c^u
m	0	$-N_b^w$	$-N_c^w$

Hence, the Ahagon, Ren, and Yioultsis spaces are the same (they are also known to coincide with Nedelec's space).

Likewise, the union of Ahagon $\cup$ Kameari spaces has the same dimension (20) as the Ahagon space alone; hence, the Ahagon space is a subspace of Kameari's.

In a similar way, one verifies that Webb's space coincides with Lee's (this can also be seen directly from the definition of these spaces). At the same time, the mutual mass matrix of the Webb and Ren bases turns out to have the nullspace of dimension 16 (rather than 20), i.e., the union of Webb $\cup$ Ren spaces has $24\ (= 40 - 16)$ linearly independent functions. Using the idea expressed in (4) and (5), one finds four functions of the Webb space that are *not* in the Ren space: $\lambda_i \nabla(\lambda_{i+1}\lambda_{i+2})$ ($i = 1, 2, 3, 4$, addition modulo 4). Similar results can be inferred from Kameari's paper [4] but are explicitly expressed there for triangular rather than tetrahedral elements (see [4, Fig. 1]).

Remark 2: The fact that the Webb basis is hierarchical is noteworthy but not dealt with here.

Curl-matrix analysis shows that not only the dimension of the curl-space for each of the element families considered above is 11, but also all pairwise intersections of these curl-spaces have the same dimension 11. This implies that all curl-spaces are simply the same. The irrotational subspaces have dimension $13\ (= 24 - 11)$ for Kameari's elements and dimension $9\ (= 20 - 11)$ for all other elements. (The irrotational subspaces are in fact spanned by all nine linearly independent gradients of quadratic functions of coordinates plus, for Kameari's elements, four gradients $\nabla(\lambda_i \lambda_{i+1} \lambda_{i+2})$.)

Approximate condition numbers of the mass and curl matrices can also be easily found by symbolic algebra[3] [21]. However, the issue of conditioning is only of tangential interest in the context of this paper; it is central in [16] and [17], as well as in [6] where the influence of conditioning on the convergence of edge element iterative solvers was examined.

IV. Higher Order Elements

The general methodology of symbolic algebra analysis of course is applicable to elements of any order and is limited only by the CPU time and memory. Although a systematic study of this kind has not yet been carried out for higher order elements, some sample computations show that such analysis is quite feasible. As a representative example, it takes approximately 5 min on an 800-MHz Pentium PC to compute the mutual mass matrix of the Graglia–Wilton–Peterson basis [10] with 45 functions (GWP45) and the Ren basis with 20 functions, and another 64 min to find the nullspace of that matrix (confirming that the

[1] The Macsyma package by Macsyma, Inc. was used.

[2] To identify each family of elements, in many cases only one of the authors' names is used for the sake of brevity, with apologies to the other authors. See Appendix and References for all names.

[3] Clearly, for curl matrices conditioning should be analyzed in the subspace orthogonal to the nullspace.

Ren20 space is a subspace of GWP45). The CPU time to form the mutual *curl* matrix of the same bases is 35 min (no attempt has been made to optimize the symbolic code).

Elements that are difficult to analyze by symbolic algebra (say, with about a hundred or more basis functions) are arguably impractical to use in numerical FE analysis anyway.

V. Examples: R11 and B46 Elements

As further examples, we apply symbolic algebra to rectangular elements with 11 degrees of freedom (R11) and brick elements with 46 degrees of freedom (B46) introduced in [18].

If λ_i are the standard four bilinear basis functions on a rectangular element, one can consider the elementwise set of functions $\{\lambda_i \nabla \lambda_j\}, (i = 1, 2, 3, 4; j = i{+}1, i{+}2, i{+}3,$ addition modulo $4)$. The Gram matrix nullspace of these functions turns out not to be empty, namely

$$\lambda_1 \nabla \lambda_4 - \lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2 + \lambda_4 \nabla \lambda_1 = 0 \qquad (6)$$

where it is assumed that node pairs 1–4 and 2–3 form the two diagonals of the rectangle. The linear dependence (6) could be verified directly by noting that $\lambda_1 \lambda_4 = \lambda_2 \lambda_3$.

To extract a basis of 11 functions, one can replace four linearly dependent functions corresponding to the diagonals, namely, $\lambda_1 \nabla \lambda_4$, $\lambda_4 \nabla \lambda_1$, $\lambda_2 \nabla \lambda_3$, $\lambda_3 \nabla \lambda_2$, with three independent functions $\lambda_1 \nabla \lambda_4 - \lambda_4 \nabla \lambda_1$, $\lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2$, and $\nabla(\lambda_1 \lambda_4 + \lambda_2 \lambda_3)$ all of which vanish on the edges. A basis is formed by these three functions, together with the eight "edge-based" functions ($\lambda_1 \nabla \lambda_2$, $\lambda_2 \nabla \lambda_1$, $\lambda_2 \nabla \lambda_4$, etc.). The Gram matrix of these 11 functions is indeed nonsingular.

This elementwise space allows a simple description as

$$\text{span}\{\hat{x},\ \hat{x}x,\ \hat{x}y,\ \hat{x}y^2,\ \hat{x}xy,\ \hat{y},\ \hat{y}x,\ \hat{y}y,\ \hat{y}x^2,\ \hat{y}xy,\ \hat{x}xy^2 + \hat{y}x^2y\} \qquad (7)$$

Indeed, the range of the mutual Gram matrix of this basis and the original one has dimension 11, i.e., the union of both bases contains the same number of linearly independent functions (11) as each basis separately.

If the last basis function in (7) is broken up into two individual terms, one obtains the $Q_{1,2}$ element with 12 degrees of freedom. The R11 space is thus a subspace of $Q_{1,2}$ but their irrotational subspaces are the same, and R11 elements can be expected to be spurious-free [13].

Higher order rectangular elements of [18] can be examined in a similar manner. For example, the R11 space can be enriched with the additional 36 functions $\lambda_i \lambda_j \nabla \lambda_k$ ($i = 1, 2, 3, 4$; $j = i+1, i+2, i+3$, $k = j+1, j+2, j+3$, index arithmetic modulo 4) (the motivation for considering this space is given in [18]). This results in 48 tangentially continuous approximating functions, with a linearly independent subset of 23. In addition to the $Q_{2,2}$ functions, this R23 space includes $\{\hat{x}x^3y,\ \hat{x}xy^3,\ \hat{y}x^3y,\ \hat{y}xy^3,\ \hat{x}x^2y^3 + \hat{y}x^3y^2\}$. The irrotational subspace of R23 has dimension 15 and contains gradients of all polynomials in x, y up to bicubic. It can then be easily verified that the PLG condition [13] that is critical for spectral convergence in cavity resonance problems is satisfied.

Similar procedures can be applied to three-dimensional brick elements. The space spanned by 56 functions $\{\lambda_i \nabla \lambda_j\}$, $i = 1, 2, \ldots, 8$; $j = i+1, i+2, \ldots, i+7$ (addition modulo 8), where the λ's are now the standard trilinear nodal functions, has the actual dimension of 46. (There are six linear dependencies, one per face, similar to the rectangular case, and four additional dependencies that can be traced to the "big diagonals.")

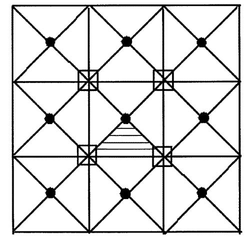

Fig. 1. Criss-cross mesh with 36 triangular elements.

VI. Prolongation of Gradients and Criss-Cross Meshes

The companion paper [13] introduces PLG, a new condition for spectral convergence. PLG requires that any irrotational (gradient) field over any individual element of the mesh can be extended to a valid discrete irrotational field over the whole mesh, with zero tangential components at the boundary (see [13] for details).

With this in mind, symbolic algebra has been applied to the curious, and somewhat controversial, case of crisscross meshes. Consider such a mesh with 3 $\times$ 3 layers of elements, 36 triangular elements altogether (Fig. 1). The small size of the mesh does not in fact reduce the generality of analysis because PLG must apply to all elements, including the ones that are only one layer away from the domain boundary (see [13, Remark 1]).

Vector fields are approximated componentwise on first order nodal elements. There are 17 degrees of freedom altogether (two per node marked with a solid circle or a square, plus one normal component of the field for the eight nodes on the boundary, except the corners). The 17 $\times$ 17 curl-matrix has the nullspace of dimension eight (this can also be inferred from [19]—in general, the dimension is equal to the number of "solid circle nodes" minus one [11]). The irrotational space over a single element (say, the shaded triangle in Fig. 1) is comprised of gradients of all quadratic functions of x and y, and thus has dimension five. Direct symbolic algebra computation then confirms that each of the five independent gradients actually lies in the global irrotational space, so the PLG condition holds. Nevertheless crisscross elements are not spurious-free [11] because other necessary conditions of spectral convergence are violated [13], [20].

VII. Conclusion

The paper advocates wider use of symbolic algebra in edge element analysis for determining inclusions and intersections of FE spaces and of their curl's; irrotational subspaces; extraction of bases; prolongation of gradients, and other tasks. The analysis provides some reference material on tetrahedral edge element families and some novel rectangular and hexahedral tangentially continuous elements. The PLG condition essential for spectral convergence is also analyzed by symbolic algebra.

APPENDIX
BASES FOR TETRAHEDRAL EDGE ELEMENT FAMILIES

The list below is representative but not exhaustive—it is impossible to include all existing families of elements.

λ_i is the barycentric coordinate corresponding to node i ($i = 1, 2, 3, 4$).

1) The Ahagon–Kashimoto basis (20 functions) [3].

$$\{12 \text{ ``edge'' functions } (4\lambda_i - 1)(\lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i),\ i \neq j\} \cup \\ 4\lambda_1(\lambda_2\nabla\lambda_3 - \lambda_3\nabla\lambda_2),\ 4\lambda_2(\lambda_3\nabla\lambda_1 - \lambda_1\nabla\lambda_3), \\ 4\lambda_1(\lambda_3\nabla\lambda_4 - \lambda_4\nabla\lambda_3),\ 4\lambda_4(\lambda_1\nabla\lambda_3 - \lambda_3\nabla\lambda_1), \\ 4\lambda_1(\lambda_2\nabla\lambda_4 - \lambda_4\nabla\lambda_2),\ 4\lambda_2(\lambda_1\nabla\lambda_4 - \lambda_4\nabla\lambda_1), \\ 4\lambda_2(\lambda_3\nabla\lambda_4 - \lambda_4\nabla\lambda_3),\ 4\lambda_4(\lambda_2\nabla\lambda_3 - \lambda_3\nabla\lambda_2).$$

2) The Kameari basis (24 functions) [4].

$$\{\text{the Lee basis}\} \cup \{\nabla(\lambda_2\lambda_3\lambda_4),\ \nabla(\lambda_1\lambda_3\lambda_4), \\ \nabla(\lambda_1\lambda_2\lambda_4),\ \nabla(\lambda_1\lambda_2\lambda_3)\}.$$

3) The Lee–Sun–Cendes basis (20 functions) [5].

$$\{12 \text{ edge-based functions } \lambda_i\nabla\lambda_j \quad i \neq j\} \cup \\ \{\lambda_1\lambda_2\nabla\lambda_3,\ \lambda_1\lambda_3\nabla\lambda_2,\ \lambda_2\lambda_3\nabla\lambda_4,\ \lambda_2\lambda_4\nabla\lambda_3,\ \lambda_3\lambda_4\nabla\lambda_1, \\ \lambda_3\lambda_1\nabla\lambda_4,\ \lambda_4\lambda_1\nabla\lambda_2,\ \lambda_4\lambda_2\nabla\lambda_1\}.$$

4) The Ren–Ida basis (20 functions) [6].

$$\{12 \text{ edge-based functions } \lambda_i\nabla\lambda_j \quad i \neq j\} \cup \\ \{\lambda_1\lambda_2\nabla\lambda_3 - \lambda_2\lambda_3\nabla\lambda_1,\ \lambda_1\lambda_3\nabla\lambda_2 - \lambda_2\lambda_3\nabla\lambda_1, \\ \lambda_1\lambda_2\nabla\lambda_4 - \lambda_4\lambda_2\nabla\lambda_1,\ \lambda_1\lambda_4\nabla\lambda_2 - \lambda_4\lambda_2\nabla\lambda_1, \\ \lambda_1\lambda_3\nabla\lambda_4 - \lambda_4\lambda_3\nabla\lambda_1,\ \lambda_1\lambda_4\nabla\lambda_3 - \lambda_3\lambda_4\nabla\lambda_1, \\ \lambda_2\lambda_3\nabla\lambda_4 - \lambda_4\lambda_3\nabla\lambda_2,\ \lambda_2\lambda_4\nabla\lambda_3 - \lambda_3\lambda_2\nabla\lambda_4\}.$$

5) The Savage–Peterson basis [7].

$$\{12 \text{ edge-based functions } \lambda_i\nabla\lambda_j \quad i \neq j\} \cup \{\lambda_i\lambda_j\nabla\lambda_k \\ -\lambda_i\lambda_k\nabla\lambda_j,\ \lambda_i\lambda_j\nabla\lambda_k - \lambda_j\lambda_k\nabla\lambda_i,\ 1 \leq i < j < k \leq 4\}.$$

6) The Yioultsis–Tsiboukis basis (20 functions) [8].

$$\{\left(8\lambda_i^2 - 4\lambda_i\right)\nabla\lambda_j + (-8\lambda_i\lambda_j + 2\lambda_j)\nabla\lambda_i,\ i \neq j\} \cup \\ \{16\lambda_1\lambda_2\nabla\lambda_3 - 8\lambda_2\lambda_3\nabla\lambda_1 - 8\lambda_3\lambda_1\nabla\lambda_2;\ 16\lambda_1\lambda_3\nabla\lambda_2 \\ - 8\lambda_3\lambda_2\nabla\lambda_1 - 8\lambda_2\lambda_1\nabla\lambda_3;\ 16\lambda_4\lambda_1\nabla\lambda_2 - 8\lambda_1\lambda_2\nabla\lambda_4 \\ - 8\lambda_2\lambda_4\nabla\lambda_1;\ 16\lambda_4\lambda_2\nabla\lambda_1 - 8\lambda_2\lambda_1\nabla\lambda_4 - 8\lambda_1\lambda_4\nabla\lambda_2; \\ 16\lambda_2\lambda_3\nabla\lambda_4 - 8\lambda_3\lambda_4\nabla\lambda_2 - 8\lambda_4\lambda_2\nabla\lambda_3;\ 16\lambda_2\lambda_4\nabla\lambda_3 \\ - 8\lambda_4\lambda_3\nabla\lambda_2 - 8\lambda_3\lambda_2\nabla\lambda_4;\ 16\lambda_3\lambda_1\nabla\lambda_4 - 8\lambda_1\lambda_4\nabla\lambda_3 \\ - 8\lambda_4\lambda_3\nabla\lambda_1;\ 16\lambda_3\lambda_4\nabla\lambda_1 - 8\lambda_4\lambda_1\nabla\lambda_3 - 8\lambda_1\lambda_3\nabla\lambda_4\}.$$

7) The Webb–Forghani basis (20 functions) [9].

$$\{6 \text{ edge-based functions } \lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i,\ i \neq j\} \cup \\ \{6 \text{ edge-based functions } \nabla(\lambda_i\lambda_j)\ i \neq j\} \cup \\ \{\lambda_1\lambda_2\nabla\lambda_3,\ \lambda_1\lambda_3\nabla\lambda_2,\ \lambda_2\lambda_3\nabla\lambda_4,\ \lambda_2\lambda_4\nabla\lambda_3,\ \lambda_3\lambda_4\nabla\lambda_1, \\ \lambda_3\lambda_1\nabla\lambda_4,\ \lambda_4\lambda_1\nabla\lambda_2,\ \lambda_4\lambda_2\nabla\lambda_1\}.$$

8) The Graglia–Wilton–Peterson basis (20 functions) [10].

$$\{(3\lambda_i - 1)(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i),\ i \neq j\} \cup \\ \frac{9}{2} \times \{\lambda_2(\lambda_3\nabla\lambda_4 - \lambda_4\nabla\lambda_3),\ \lambda_3(\lambda_4\nabla\lambda_2 - \lambda_2\nabla\lambda_4), \\ \lambda_3(\lambda_4\nabla\lambda_1 - \lambda_1\nabla\lambda_4),\ \lambda_4(\lambda_1\nabla\lambda_3 - \lambda_3\nabla\lambda_1), \\ \lambda_4(\lambda_1\nabla\lambda_2 - \lambda_2\nabla\lambda_1),\ \lambda_1(\lambda_4\nabla\lambda_2 - \lambda_2\nabla\lambda_4), \\ \lambda_1(\lambda_2\nabla\lambda_3 - \lambda_3\nabla\lambda_2),\ \lambda_2(\lambda_1\nabla\lambda_3 - \lambda_3\nabla\lambda_1)\}.$$

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