



# Finite element differential–algebraic systems for eddy current problems \*

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Finite element discretization of some time-dependent eddy current problems yields ordinary differential–algebraic systems with large sparse matrices. Properties and stability of these systems are analyzed for two classes of eddy current problems: (i) two-dimensional coupled field-circuit problems with arbitrary external circuit connections between conductors; (ii) “2.5-dimensional” problems characterized by axisymmetric geometry and non-axisymmetric excitation. Extension of the analysis to many other formulations of eddy current problems in 2D and 3D is straightforward.

**Keywords:** finite elements, stability, differential–algebraic systems, eddy currents

**AMS subject classification:** 65M12, 65M60

## 1. Introduction

Eddy current problems are described by the Maxwell equations with the displacement currents neglected. In engineering applications (such as electric machines) the computational domain typically contains large nonconducting regions (“air”) where the field distribution is essentially quasi-static and can be described by elliptic equations for a magnetic potential. Dynamic effects are present only in conductors. Numerical discretization of such problems by Finite Difference, Finite Element (FE) or other methods leads in a natural way to Differential–Algebraic (DA) systems, the differential part corresponding to conductors, and the algebraic part – to nonconductors. It is the objective of this paper to study stability of such systems and properties of their block matrices: positive (semi)definiteness, the index of nilpotency, and the structure of null spaces. The focus is on the FE-Galerkin method: being a restriction of the weak formulation of the continuous problem to a finite-dimensional subspace, it preserves the essential properties of the continuous problem.

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Numerous formulations exist for two-dimensional, ‘2.5’-dimensional (see below), and three-dimensional eddy current problems. The differences lie in the way vector/scalar potentials and gauges are introduced, and in the treatment of boundary conditions. While it is not feasible to cover all of these formulations in one paper, the approach taken here for two particular cases can certainly be extended to other situations.

These two cases are as follows:

1. A coupled field-circuit problem in 2D. Massive and stranded (filamentary) conductors with eddy currents are connected by arbitrary external circuits.
2. A ‘2.5D’ problem where the geometry is axisymmetric but the field sources are not. Field quantities can be expanded into the Fourier series with respect to the angle.

Although both of these problems have been extensively studied in the electromagnetic literature [2,3,7,8,10–13], almost no formal attention has been paid to the properties of the resultant DA systems. For the coupled 2D problem, this paper expands on the stability analysis of [13] and provides relevant rigorous proofs in section 3. Stability analysis of the ‘2.5D’ problem appears in section 4.5.

## 2. Formulation of the coupled field-circuit problem in 2D

### 2.1. Field-circuit differential equations

The coupled field-circuit problem consists in finding the distribution of the electromagnetic field and eddy currents generated by multiple conductors connected at their ends by external circuits (figure 1), which is typical in the analysis of electric machines. The conductors are assumed long enough, so that the problem can be treated as essentially two-dimensional.

The computational domain  $\Omega$  may contain conductors of three types: (1) massive (solid) conductors  $\Omega_k^{(m)}$ ; (2) stranded (filamentary) conductors  $\Omega_k^{(s)}$ ; (3) conductors with known current densities. Let the number of massive and stranded conductors be  $n_{\text{mass}}$  and  $n_s$ , respectively.

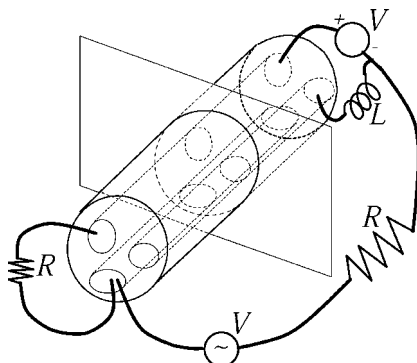


Figure 1. A generic field-circuit problem.

Electromagnetic fields in the central cross-section of the machine are usually described in terms of the one-component magnetic vector potential  $A$  and the electric scalar potential gradient  $g$ . This gradient is constant within the cross-section of each conductor (see, e.g., [10]).

A complete electromagnetic formulation of this coupled problem is derived in [11,12]. The circuit mesh (loop) current method is used to describe external circuit connections between stranded and massive conductors.

The formulation consists of three groups of equations: field equations, integral conditions for total currents in massive conductors, and circuit mesh equations. Both vectors  $A$  and  $g$  have only  $z$ -components and can therefore be treated as scalars. Within the  $k$ th massive conductor  $\Omega_k^{(m)}$ ,  $g = g_k = \text{const}$ . Field equations in massive conductors  $\Omega_k^{(m)}$ , stranded conductors  $\Omega_k^{(s)}$  and conductors  $\Omega_0$  with a given current density are as follows:

### Field equations

$$-\nabla \cdot \nu \nabla A + \sigma (\partial_t A + g_k) = 0 \quad \text{in } \Omega_k^{(m)}, \quad k = 1, 2, \dots, n_{\text{mass}} \quad (1)$$

$$-\nabla \cdot \nu \nabla A = \frac{(P^{(s)}i)_k}{S_k} \quad \text{in } \Omega_k^{(s)}, \quad k = 1, 2, \dots, n_s, \quad (2)$$

$$-\nabla \cdot \nu \nabla A = J_0(x, y) \quad \text{in } \Omega^{(0)}, \quad (3)$$

$$\nu = \nu(x, y) \geq \nu_0 > 0, \quad (4)$$

$$\sigma = \sigma(x, y) \geq \sigma_0 > 0 \quad \text{in } \Omega_k^{(m)}; \quad \sigma = 0 \quad \text{in } \Omega \setminus \bigcup \overline{\Omega_k^{(m)}}. \quad (5)$$

**Total currents** in massive (solid) conductors:<sup>1</sup>

$$\int_{\Omega_k^{(m)}} \sigma \partial_t A \, dS + g_k \int_{\Omega_k^{(m)}} \sigma \, dS + (P^{(m)}i)_k = 0. \quad (6)$$

**Kirchoff equations** for circuit mesh currents:

$$-P^{(s)T} \left\{ \frac{1}{S_k} \int_{\Omega_k^{(s)}} \partial_t A \, dS \right\} + P^{(m)T} g - \widehat{R}i - \widehat{M}d_t i - P^{(s)T} R^{(s)} P^{(s)}i = \widehat{v}_{\text{src}} \quad (7)$$

where the angle brackets  $\{ \}$  denote a column matrix with given entries;  $R^{(s)}$  is the diagonal matrix composed of the dc resistances  $R_k^{(s)}$ ;  $\widehat{R} = R/l_{\text{eff}}$ ;  $\widehat{M} = M/l_{\text{eff}}$ ;  $\widehat{v}_{\text{src}} = v_{\text{src}}/l_{\text{eff}}$ ;  $R$  and  $M$  are symmetric positive definite matrices of mutual resistances and inductances between the circuit meshes;  $l_{\text{eff}}$  is the effective length of conductors (assumed to be identical for all conductors in the model);  $v_{\text{src}}$  is a column matrix of sums of voltages of given voltage sources in the circuit meshes. Finally,  $P^{(m)}$  and  $P^{(s)}$  are the connectivity matrices (characterizing the topology of external circuits), with elements 0 or  $\pm 1$  indicating which conductors (massive and stranded, respectively) belong to which circuit meshes [10,11].

<sup>1</sup> This category also includes stranded (filamentary) conductors if the filaments are *not* transposed (twisted) along their length.

Even though in engineering problems magnetic materials are usually nonlinear, to simplify the analysis and to concentrate on the most salient features of the DA system, only linear problems will be considered in the paper. In addition, the continuous problem will not be studied here and some standard items (such as initial and boundary conditions, smoothness requirements, etc.) will be omitted in order not to obscure the main subject of this analysis.

## 2.2. Finite element discretization

None of the main results and conclusions of the paper are affected in a critical way by the particular type of FE discretization chosen. However, for ease of reference and for definiteness, we shall sometimes refer to the standard first order triangular elements. Let the total number of 'active' (non-Dirichlet) FE degrees of freedom (i.e. nodes in case of Lagrangian elements) be  $n$ , and the number of degrees of freedom strictly outside the conducting subregions be  $n_{\text{dof}}^{\text{air}}$ .

Further, let  $n_{\text{dof}}^{\text{mass}}$  and  $n_{\text{dof}}^{\text{s}}$  stand for the number of FE degrees of freedom in massive and stranded conductors, respectively (including their boundaries<sup>2</sup>).

For illustration, one actual simulation example is given in figure 2, where magnetic field lines in a large synchronous motor are shown.<sup>3</sup>

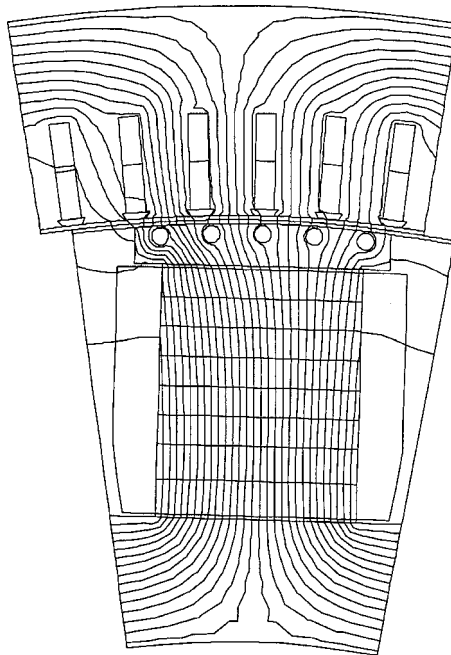


Figure 2. Flux lines in a large synchronous motor.

<sup>2</sup> We assume that separate conductors do not abut, which is normally the case in practice.

<sup>3</sup> Simulations were performed jointly with General Electric Canada, Inc. See [12] for details.

### 2.3. The Galerkin formulation of the coupled problem

The weak formulation of equations (1)–(3) is standard and can be written as follows:

$$\begin{aligned} (\nu \nabla A, \nabla A')_{L_2(\Omega)^2} + (\sigma \partial_t A, A')_{L_2(\Omega)} + (\sigma g, A')_{L_2(\Omega)} &= (J, A')_{L_2(\Omega)}, \\ A &\in H_0^1(\Omega), \quad \forall A' \in H_0^1(\Omega), \\ g &= \sum_{k=1}^{n_{\text{mass}}} g_k, \quad g_k = \text{const}(\Omega_k^{(m)}), \quad g_k = 0 \quad \text{in } \Omega \setminus \bigcup \Omega_k^{(m)}, \end{aligned} \quad (8)$$

where  $J$  stands for one of the right-hand sides of (1)–(3), and  $H_0^1(\Omega)$  is the standard Sobolev space, with Dirichlet conditions on at least part of the boundary  $\partial\Omega$  taken into account. A discrete formulation is obtained by restricting  $A$  and  $A'$  to a finite dimensional subspace  $W$  of  $H_0^1(\Omega)$  (e.g., the subspace of continuous functions linear within each of the elements of a triangular mesh and satisfying essential boundary conditions). The inner products of (8) become bilinear forms in  $W \times W$ . A choice of a basis in  $W$  induces an isomorphism between  $W$  and  $\mathbb{R}^n$  – a one-to-one correspondence between functions  $A$  in  $W$  and vectors  $a$  in  $\mathbb{R}^n$ , – as well as matrices  $Q$ ,  $L$ , and  $B_m$  such that

$$(\nu \nabla A, \nabla A')_{L_2(\Omega)^2} = (La, a')_{\mathbb{R}^n}, \quad (9)$$

$$(\sigma A, A')_{L_2(\Omega)} = (Qa, a')_{\mathbb{R}^n}, \quad (10)$$

$$(\sigma g, A')_{L_2(\Omega)} = (B_m g, a')_{\mathbb{R}^n}. \quad (11)$$

(A slight abuse of notation in the last expression and a few similar ones in the remainder of the paper: depending on the context,  $g$  means either an  $\mathbb{R}^{n_{\text{mass}}}$  vector of (constant) scalar potentials in massive conductors or an  $L_2$ -function that is constant within each massive conductor and zero elsewhere.)

Let  $A_k \in H_0^1(\Omega)$  be a magnetic potential that is identically equal to one within the  $k$ th massive conductor, is zero in all other conductors, and whose values elsewhere are arbitrary (and unimportant). The FE vector in  $\mathbb{R}^n$  corresponding to any such function will be denoted with  $\bar{1}_k$ .

The following properties of matrix blocks either constitute definitions of these blocks or are direct consequences of the definitions:

$$\int_{\Omega_k^{(m)}} \sigma A \, d\Omega = (B_m^T a)_k, \quad (12)$$

$$\int_{\Omega_k^{(m)}} \sigma \, d\Omega = S_k, \quad S = \text{diag}\{S_k\}, \quad (13)$$

$$B_m^T \bar{1}_k = S_k, \quad (14)$$

$$Q \bar{1}_k = B_m \hat{g}^{(k)}, \quad \text{where } \hat{g}_k^{(k)} = 1, \hat{g}_i^{(k)} = 0, i \neq k, \quad (15)$$

with  $k = 1, 2, \dots, n_{\text{mass}}$  in all of the expressions above. Identity (15) holds because, for any vector  $a' \in \mathbb{R}^n$ ,

$$(Q\bar{1}_k, a')_{\mathbb{R}^n} = (\sigma A_k, A')_{L_2(\Omega)} = \int_{\Omega_k^{(m)}} \sigma A' \, d\Omega = (B_m \hat{g}^{(k)}, a')_{\mathbb{R}^n}. \quad (16)$$

#### 2.4. Matrix properties of the differential–algebraic system

The DA system for the coupled field-circuit problem (1)–(3), (6), (7) has the form

$$U d_t y + V y = f(t) \quad (17)$$

where ([11], with slight notational changes)

$$U = \begin{pmatrix} Q & 0 & 0 \\ B_m^T & 0 & 0 \\ B_s^T & 0 & -M \end{pmatrix}, \quad (18)$$

$$V = \begin{pmatrix} L & B_m & B_s \\ 0 & S & P \\ 0 & P^T & -R \end{pmatrix}, \quad (19)$$

and the individual matrix blocks are defined in the previous section. Obviously,  $U$  is singular.

We now turn to the algebraic properties of the system above. We shall frequently need the following well-known factorization of a generic block matrix  $K$ :

$$K \equiv \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} = \begin{pmatrix} I & K_{12}K_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{K}_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \quad (20)$$

$$\implies \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = |K_{22}| |\tilde{K}_{11}| \quad (21)$$

where  $\tilde{K}_{11}$  is the Schur complement

$$\tilde{K}_{11} = K_{11} - K_{12}K_{22}^{-1}K_{21}. \quad (22)$$

The theorems below follow directly from the fact that the discrete problem is obtained from the weak formulation of the continuous one by simply restricting the bilinear forms to a finite-dimensional subspace; hence the main properties of the continuous problem are inherited by the discrete one.

**Theorem 1.** Matrices  $Q$ ,  $L$ ,  $S$  are symmetric positive definite.

*Proof.* Follows immediately from (9), (10), (13).  $\square$

**Theorem 2.** Matrix  $V$  (19) is nonsingular.

*Proof.* Since  $L$  is nonsingular (even symmetric positive definite by (9)), one only needs to show that  $\begin{pmatrix} S & P \\ P^T & -R \end{pmatrix}$  is nonsingular. But this is true because  $S$  and  $R$  are symmetric positive definite and therefore the Schur complement  $\tilde{S} = S + PR^{-1}P^T$  is symmetric positive definite, too.  $\square$

**Theorem 3.** Matrix

$$Y = \begin{pmatrix} Q & B_m \\ B_m^T & S \end{pmatrix} \quad (23)$$

is symmetric nonnegative definite.

*Proof.* By construction of FEM (9)–(11), (13), one has

$$\begin{aligned} & \left( Y \begin{pmatrix} a \\ g \end{pmatrix}, \begin{pmatrix} a \\ g \end{pmatrix} \right)_{E^n + n_{\text{mass}}} \\ &= (Qa, a)_{E^n} + (B_m g, a)_{E^n} + (B_m^T a, g)_{E^{n_{\text{mass}}}} + (Sg, g)_{E^{n_{\text{mass}}}} \\ &= (\sigma A, A)_{L_2(\Omega)} + (\sigma g, A)_{L_2(\Omega)} + (\sigma A, g)_{L_2(\Omega)} + (\sigma g, g)_{L_2(\Omega)} \\ &= (\sigma(A + g), A + g)_{L_2(\Omega)} \geq 0. \end{aligned} \quad (24)$$

$\square$

**Theorem 4.** The null space of  $Y$  (23) is spanned by the following vectors: (a) any  $n_{\text{dof}}^{\text{air}}$  linearly independent vectors corresponding to functions  $A$  and  $g$  that vanish within all massive conductors; and (b)  $n_{\text{mass}}$  vectors corresponding to functions  $A = -g_k = 1$  within the  $k$ th massive conductor ( $k = 1, 2, \dots, n_{\text{mass}}$ ) and zero elsewhere.

*Proof.* (a) follows from the fact that  $\sigma = 0$  outside massive conductors and (b) follows from (24).  $\square$

### 3. Eigenvalue analysis of the coupled problem

#### 3.1. The general field-circuit case

The behavior of DA systems of the generic form (17) has been extensively studied (see, e.g., [1,5,6]). In our case, matrix  $U$  is singular but  $V$  is nonsingular, which simplifies the analysis. Transforming the matrix pencil  $\lambda V + U$ , one obtains:

$$\begin{aligned} \lambda V + U &= V(\lambda I + V^{-1}U) = VT^{-1} \left[ \lambda I + \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix} \right] T \\ &= VT^{-1} \begin{pmatrix} J_1^{-1} & 0 \\ 0 & I \end{pmatrix} \left[ \lambda \begin{pmatrix} J_1 & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & J_0 \end{pmatrix} \right] T. \end{aligned} \quad (25)$$

Here  $J_0, J_1$  are the Jordan blocks of  $V^{-1}U$  corresponding to its zero and nonzero eigenvalues, respectively;  $T$  is the matrix transforming  $V^{-1}U$  to the Jordan form. The square brackets in (25) are the Kronecker Canonical Form of the matrix pencil [4,5]:

$$\text{KCF: } \lambda \begin{pmatrix} J_1 & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & J_0 \end{pmatrix}. \quad (26)$$

Thus the KCF contains the Jordan blocks of  $V^{-1}U$ . It is well known that the index of nilpotency of  $J_0$  plays a key role in the stability of the DA system and strongly influences the choice of time-stepping algorithms [1,5,6].

Properties of (17) are thus determined by the Jordan form of  $V^{-1}U$ . Therefore we consider the eigenvalue problem

$$V^{-1}Uz = \lambda z, \quad \text{or} \quad Uz = \lambda Vz. \quad (27)$$

The eigenvalues satisfy

$$\begin{vmatrix} Q - \lambda L & -\lambda B_m & -\lambda B_s \\ B_m^T & -\lambda S & -\lambda P \\ B_s^T & -\lambda P^T & -M + \lambda R \end{vmatrix} = 0. \quad (28)$$

**Theorem 5** [13]. All eigenvalues of (28) are real nonnegative.

*Proof.* Division of the second block column of (28) by  $-\lambda$  (for  $\lambda \neq 0$ ) yields a symmetric matrix, hence the eigenvalues are real. If there is an eigenvalue  $\lambda < 0$ , there exists a nonzero vector  $x = (a, g, i)^T$  such that

$$\begin{pmatrix} Q - \lambda L & B_m & B_s \\ B_m^T & S & P \\ B_s^T & P^T & \frac{1}{\lambda}M - R \end{pmatrix} \begin{pmatrix} a \\ g \\ i \end{pmatrix} = 0. \quad (29)$$

Eliminating  $i$  leads to the Schur complement equation

$$\left[ \begin{pmatrix} Q - \lambda L & B_m \\ B_m^T & S \end{pmatrix} + \begin{pmatrix} B_s \\ P \end{pmatrix} (-\lambda^{-1}M + R)^{-1} (B_s^T \ P^T) \right] \begin{pmatrix} a \\ g \end{pmatrix} = 0. \quad (30)$$

For  $\lambda < 0$ , comparison of the matrix in the square brackets of (30) with the positive semidefinite matrix  $Y$  (23) shows that the matrix in (30) is positive definite, and therefore both  $a$  and  $g$  must be zero. Then, as follows from (29),  $i$  is also zero.  $\square$

Let us now consider the eigenvectors for two cases:  $\lambda > 0$  and  $\lambda = 0$ .

(1)  $\lambda > 0$

$$\begin{pmatrix} Q - \lambda L & -\lambda B_m & -\lambda B_s \\ B_m^T & -\lambda S & -\lambda P \\ B_s^T & -\lambda P^T & -M + \lambda R \end{pmatrix} \begin{pmatrix} a \\ g \\ i \end{pmatrix} = 0$$

or, introducing  $\tilde{g} = -\lambda g$  and  $\tilde{i} = -\lambda i$ ,

$$\begin{pmatrix} Q - \lambda L & B_m & B_s \\ B_m^T & S & P \\ B_s^T & P^T & \lambda^{-1}M - R \end{pmatrix} \begin{pmatrix} a \\ \tilde{g} \\ \tilde{i} \end{pmatrix} = 0.$$

This matrix is symmetric; hence there is a full set of eigenvectors corresponding to each positive eigenvalue  $\lambda$ . Scaling back to  $(a, g, i)$ , we obtain a full set of eigenvectors for the original problem.

(2)  $\lambda = 0$ . To determine the multiplicity of  $\lambda = 0$ , we need to consider the block matrix structure of (28) more carefully. There are five categories of unknowns:

Category of unknowns	Notation	Number of unknowns
node values of $A$ in nonconducting regions	$a_{\text{air}}$	$n_{\text{dof}}^{\text{air}}$
node values of $A$ in massive conductors	$a_{\text{mass}}$	$n_{\text{dof}}^{\text{mass}}$
node values of $A$ in stranded conductors	$a_s$	$n_{\text{dof}}^s$
scalar potential gradients	$g_k$	$n_{\text{mass}}$
circuit loop currents	$i$	$n_{\text{loop}}$

The matrix in (28) can be partitioned accordingly,<sup>4</sup> as a  $5 \times 5$  block matrix:

	$a_{\text{air}}$	$a_{\text{mass}}$	$a_s$	$g_k$	$i$
$a_{\text{air}}$	$-\lambda L_{aa}$	$-\lambda L_{am}$	$-\lambda L_{as}$	0	0
$a_{\text{mass}}$	$-\lambda L_{ma}$	$\tilde{Q} - \lambda L_{mm}$	$-\lambda L_{ms}$	$-\lambda \tilde{B}_m$	0
$a_s$	$-\lambda L_{sa}$	$-\lambda L_{sm}$	$-\lambda L_{ss}$	0	$-\lambda \tilde{B}_s$
$g_k$	0	$\tilde{B}_m^T$	0	$-\lambda S$	$-\lambda P$
$i$	0	0	$\tilde{B}_s^T$	$-\lambda P^T$	$-M + \lambda R$

where the tildes denote the nonzero submatrices of  $Q$ ,  $B_m$  and  $B_s$  (that is, the submatrices corresponding to FE degrees of freedom in massive and stranded conductors, respectively).

The determinant of the above matrix can be decomposed as (21), with  $K_{22} \equiv -M + \lambda R$ , and then  $(-\lambda)$  can be factored out of the  $n_0 = n_{\text{dof}}^{\text{air}} + n_{\text{dof}}^s + n_{\text{mass}}$  columns of the respective Schur complement. Once this is done, it is straightforward to verify, with theorem 3 in mind, that the resultant matrix is nonsingular at  $\lambda = 0$ . Thus the multiplicity of  $\lambda = 0$  is  $n_0$ . The eigenvectors corresponding to  $\lambda = 0$  satisfy

$$\begin{pmatrix} Q & 0 & 0 \\ B_m^T & 0 & 0 \\ B_s^T & 0 & -M \end{pmatrix} \begin{pmatrix} a \\ g \\ i \end{pmatrix} = 0.$$

There are exactly  $n_0$  linearly independent null eigenvectors: (a)  $n_{\text{mass}}$  eigenvectors with  $g_k = 1$  ( $k = 1, 2, \dots, n_{\text{mass}}$ ) and all other entries equal to zero; (b)  $n_{\text{dof}}^{\text{air}}$  eigenvectors with one nodal value of  $a$  equal to one in the air, and all other entries equal to zero; (c)  $n_{\text{dof}}^s$  vectors  $i = M^{-1} \tilde{B}_s^T a_s$ , with arbitrary but linearly independent  $a_s$ .

<sup>4</sup> This partitioning of course implies a certain numbering of the FE degrees of freedom.

Thus the number of eigenvectors is equal to the multiplicity of  $\lambda = 0$  in (28), and therefore matrix  $V^{-1}U$  is nondefective. However, as demonstrated in the following section, the situation is different for the truncated problem with given total currents and with no explicit electric circuits present.

### 3.2. A particular case: Known total currents

In the special case when external circuits contain only current sources supplying given total currents in massive conductors, and no stranded conductors, system (17)–(19) gets truncated to [11,12]

$$\begin{pmatrix} Q & 0 \\ B_m^T & 0 \end{pmatrix} d_t \begin{pmatrix} a \\ g \end{pmatrix} + \begin{pmatrix} L & B_m \\ 0 & S \end{pmatrix} \begin{pmatrix} a \\ g \end{pmatrix} = \begin{pmatrix} b \\ -I \end{pmatrix} \quad (31)$$

where matrices  $L$ ,  $Q$ ,  $B_m$  correspond to the three respective terms in (1), and the entry  $S_k$  of the diagonal matrix  $S$  corresponds to  $\int \sigma g_k dS$  in (6). As before, vector  $a$  contains the node values of  $A$ , while  $g$  is a vector of the scalar potential gradients  $g_k$ . With circuit unknowns and stranded conductor unknowns now absent, the governing determinant (28) has a natural  $3 \times 3$  partitioning:

	$a_{\text{air}}$	$a_{\text{mass}}$	$g_k$
$a_{\text{air}}$	$-\lambda L_{aa}$	$-\lambda L_{am}$	$0$
$a_{\text{mass}}$	$-\lambda L_{ma}$	$\tilde{Q} - \lambda L_{mm}$	$-\lambda \tilde{B}_m$
$g_k$	$0$	$\tilde{B}_m^T$	$-\lambda S$

After  $-\lambda$  is factored out of  $n_{\text{dof}}^{\text{air}} + n_{\text{mass}}$  columns, the resultant matrix

	$a_{\text{air}}$	$a_{\text{mass}}$	$g_k$
$a_{\text{air}}$	$L_{aa}$	$-\lambda L_{am}$	$0$
$a_{\text{mass}}$	$L_{ma}$	$\tilde{Q} - \lambda L_{mm}$	$\tilde{B}_m$
$g_k$	$0$	$\tilde{B}_m^T$	$S$

is, in contrast with the general case of section 3.1, still singular at  $\lambda = 0$ . Indeed, it has  $n_{\text{mass}}$  independent null vectors that, due to (24), correspond to  $A_k = -g_k (= \text{const})$  in each massive conductor, zero elsewhere. The multiplicity of  $\lambda = 0$  for the matrix above is thus  $2n_{\text{mass}} + n_{\text{dof}}^{\text{air}}$ . This (singular) matrix is defective; luckily, system (31) is of index one, as the following theorem shows.

**Theorem 6.** For the 2D coupled problem (31) with given total currents, the Jordan form of matrix  $V^{-1}U$  contains  $n_{\text{mass}}$  blocks  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  (i.e. one such block per massive conductor). Furthermore, matrix  $V^{-1}U$  has

- $n_{\text{mass}}$  null vectors of the form  $(0, \hat{g}_k)^T$ ;
- $n_{\text{mass}}$  vectors  $x_k$  such that

$$V^{-1}U x_k \neq 0 \quad \text{but} \quad (V^{-1}U)^2 x_k = 0 \quad (32)$$

where  $k = 1, 2, \dots, n_{\text{mass}}$  and

$$U = \begin{pmatrix} Q & 0 \\ B_m^T & 0 \end{pmatrix}, \quad V = \begin{pmatrix} L & B_m \\ 0 & S \end{pmatrix}. \quad (33)$$

*Proof.* The null vectors are verified by direct substitution. Let us show that (32) is satisfied for  $x_k \equiv (x_a, x_g)^T$ ,  $x_a = \bar{1}_k$ ,  $x_g = 0$ . Indeed,

$$Ux_k = \begin{pmatrix} Q & 0 \\ B_m^T & 0 \end{pmatrix} \begin{pmatrix} \bar{1}_k \\ 0 \end{pmatrix} = (Q\bar{1}_k, B_m^T\bar{1}_k). \quad (34)$$

Denote  $z_k \equiv (z_a, z_g)^T = V^{-1}Ux_k$ ; then

$$Vz_k = \begin{pmatrix} L & B_m \\ 0 & S \end{pmatrix} \begin{pmatrix} z_a \\ z_g \end{pmatrix}$$

and by (14), (15) this is equal to the right-hand side of (34) if  $z_a = 0$  and  $z_g = \hat{g}_k$ . But such a  $z$  is a null vector of  $U$  and hence a null vector of  $V^{-1}U$ .  $\square$

### 3.3. The time-integrated scalar potential

The DA system can be transformed to a pure differential form by a simple change of variables:  $g = d_t G$ , where the new function  $G$  is, in physical terms, gradient of the time-integrated scalar potential. (In electromagnetic *theory*, it makes little difference which potential to use, and the non-integrated one appears more frequently; however, the time-integrated potential was already mentioned in the classical book by Stratton in 1941 [9].) The analysis of the previous section can be easily adapted to this case.

*Remark 7.*<sup>5</sup> Index-1 systems are known to be rather harmless if handled properly [1,6] – e.g., by using backward differentiation formulas as in the DASSL package [1]. However, many practitioners in applied electromagnetic analysis favor the Crank–Nicholson scheme which in this case is unsuitable and leads to instability (see [13]).

## 4. The 2.5D model

### 4.1. Introduction

In the ‘2.5D’ model of the electromagnetic field, the geometry is assumed axisymmetric but the excitation currents are not. The distribution of all field quantities is expanded into a Fourier series with respect to the  $\varphi$  angle; due to the assumed linearity of the problem, each angular harmonic can be considered separately. (In practice, the main angular harmonic is typically of primary interest.)

The problem for a single harmonic is mathematically two-dimensional but at the same time preserves many essential physical features of the 3D model (e.g., vector fields retain all three components) – hence the term ‘2.5D’.

<sup>5</sup> Suggested by the anonymous reviewer.

#### 4.2. Formulation of the 2.5D problem

##### 4.2.1. Maxwell's equations

The magnetic field is described by Maxwell's equations, with displacement currents neglected:

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (35)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad (36)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (37)$$

with constitutive relationships

$$\mathbf{B} = \mu \mathbf{H}, \quad \mu = \mu(r, z) \geq \mu_0 > 0, \quad (38)$$

$$\mathbf{J} = \mathbf{J}_{\text{given}} + \sigma \mathbf{E}, \quad \sigma = \sigma(r, z) > 0. \quad (39)$$

In (39),  $\mathbf{J}_{\text{given}}$  is a known current density (typically in filamentary windings), and  $\sigma \mathbf{E}$  is the current density of induced (eddy) currents. In nonconducting regions  $\sigma = 0$ . In conductors,  $\sigma \geq \sigma_0 > 0$ . In addition, for regions with given currents  $\mathbf{J}_{\text{given}}$  we formally set  $\sigma$  to zero, to have  $\mathbf{J} = \mathbf{J}_{\text{given}}$ .

As noted in the introduction, we assume a sinusoidal variation of all fields with respect to the  $\varphi$  angle, e.g.,

$$\mathbf{B}(r, \varphi, z, t) = \hat{r} B_{kr}(r, z, t) \cos k\varphi + \hat{\varphi} B_{k\varphi}(r, z, t) \sin k\varphi + \hat{z} B_{kz}(r, z, t) \cos k\varphi, \quad (40)$$

$$\mathbf{E}(r, \varphi, z, t) = \hat{r} E_{kr}(r, z, t) \sin k\varphi + \hat{\varphi} E_{k\varphi}(r, z, t) \cos k\varphi + \hat{z} E_{kz}(r, z, t) \sin k\varphi, \quad (41)$$

$$\mathbf{J}(r, \varphi, z, t) = \hat{r} J_{kr}(r, z, t) \sin k\varphi + \hat{\varphi} J_{k\varphi}(r, z, t) \cos k\varphi + \hat{z} J_{kz}(r, z, t) \sin k\varphi, \quad (42)$$

$$u(r, \varphi, z, t) = u_k(r, z, t) \cos k\varphi \quad (43)$$

where  $\mathbf{B}$  is the magnetic flux density,  $\mathbf{E}$  is the electric field,  $\mathbf{J}$  is the current density,  $\hat{r}$ ,  $\hat{\varphi}$ ,  $\hat{z}$  are the cylindrical unit vectors, and  $k = 1, 2, \dots$  is the harmonic number.<sup>6</sup>

##### 4.2.2. The $T$ - $u$ method

In nonconducting regions, which typically take up a substantial part of the computational domain, (35) reduces to

$$\nabla \times \mathbf{H} = 0 \quad (\text{nonconducting regions}) \quad (44)$$

and therefore, assuming for simplicity that the domain is simply connected,

$$\mathbf{H} = -\nabla u \quad (\text{nonconducting regions}), \quad (45)$$

$$\mathbf{H} = \mathbf{T} - \nabla u \quad (\text{conducting regions}) \quad (46)$$

where  $\mathbf{T}$  is the vector potential of currents and is a particular solution of (35), i.e.

$$\nabla \times \mathbf{T} = \mathbf{J}. \quad (47)$$

<sup>6</sup> In engineering practice,  $k$  is actually the harmonic number times the number of pole pairs of the machine.

The difference between (47) and (35) is that  $\mathbf{T}$ , unlike  $\mathbf{H}$ , need not satisfy any additional equations or constraints. Therefore there is substantial freedom in choosing  $\mathbf{T}$ . It has now become classical that in many cases  $\mathbf{T}$  can be chosen to have only two nonzero components in a suitable coordinate system and, in addition, to be zero in nonconducting regions (see, e.g., [2,3,7] for details). For the Fourier harmonic in the angular direction, one can set

$$\mathbf{T}(r, \varphi, z, t) = \hat{r}T_{kr}(r, z, t) \cos k\varphi + \hat{z}T_{kz}(r, z, t) \cos k\varphi \quad (48)$$

the  $\varphi$ -component of the  $\mathbf{T}$  vector being zero. The two-component vector potential corresponding to given currents (in the windings) will be denoted with  $\mathbf{T}_{\text{given}}$ . Equations (36), (37) then become

$$-\nabla \cdot \mu \nabla u + \nabla \cdot \mu \mathbf{T} = -\nabla \cdot \mu \mathbf{T}_{\text{given}}, \quad (49)$$

$$-\partial_t \mu \nabla u + \nabla \times \sigma^{-1} \nabla \times \mathbf{T} + \partial_t \mu \mathbf{T} = -\partial_t \mu \mathbf{T}_{\text{given}}, \quad (50)$$

$$\mathbf{T} \times \hat{n} = 0 \quad \text{on the boundaries of conducting subregions,} \quad (51)$$

$$\mathbf{T} = 0 \quad \text{in nonconducting regions} \quad (52)$$

where  $\hat{n}$  is the unit normal vector on the conductor-air interface. Condition (51) ensures the continuity of the tangential component of  $\mathbf{H}$  on the boundaries of conductors.

In addition, the standard conditions on the exterior boundary of the computational domain must be imposed. (Typically, the tangential component of  $\mathbf{H}$  vanishes, which is equivalent to setting the magnetic scalar potential to zero. On symmetry planes, the Neumann condition for the potential is appropriate.)

In the simulation example below (figure 3), the equipotential lines ( $u = \text{const}$ ) are shown for a superconducting motor for illustration.

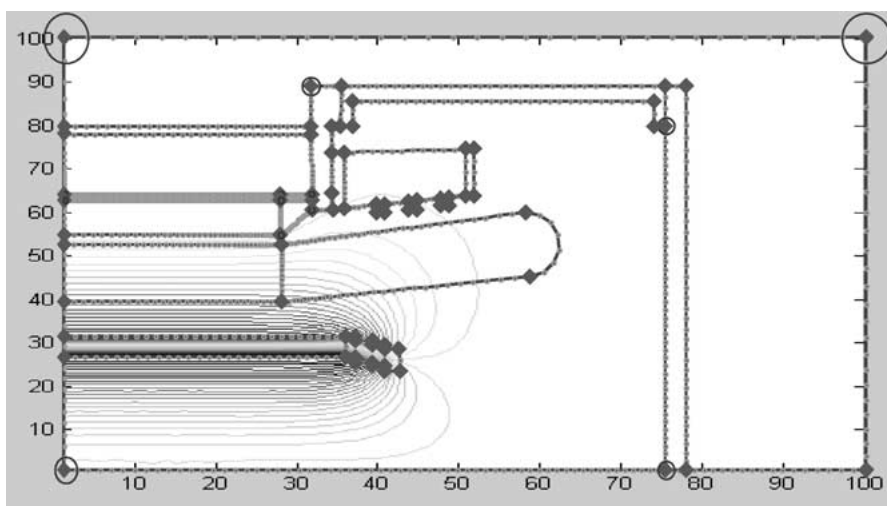


Figure 3. Scalar potential lines for a superconducting motor.

#### 4.3. The Galerkin formulation

The Galerkin formulation is obtained by the standard procedure (see, e.g., [7]). An appropriate functional space for the pair of functions  $\mathbf{T}, u$  is introduced.<sup>7</sup> Then equations (49), (50) are multiplied by a ‘trial’ pair  $\mathbf{T}', u'$  and integrated by parts. The end result is

$$(\mu \nabla u, \nabla u') - (\mu \mathbf{T}, \nabla u') = (\mu \mathbf{T}_{\text{given}}, \nabla u'), \quad (53)$$

$$(-\partial_t \mu \nabla u, \mathbf{T}') + (\partial_t \mu \mathbf{T}, \mathbf{T}') + (\sigma^{-1} \nabla \times \mathbf{T}, \nabla \times \mathbf{T}') = -\partial_t (\mu \mathbf{T}_{\text{given}}, \mathbf{T}') \quad (54)$$

where all inner products are in  $L_2(\Omega)^3$ . With a suitable set of (finite element) basis functions chosen, and with trial functions the same as basis functions (the Galerkin method), equations (53), (54) become, in the matrix form,

$$\begin{pmatrix} N & -D \\ 0 & C \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{T} \end{pmatrix} + d_t \begin{pmatrix} 0 & 0 \\ -D^T & M \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{T} \end{pmatrix} = \begin{pmatrix} \bar{f}(t) \\ -d_t \bar{g}(t) \end{pmatrix} \quad (55)$$

where the bars denote Euclidean vectors and matrix blocks correspond to continuous bilinear forms:

$$(\mu \nabla u, \nabla u')_{L_2(\Omega)^3} = (N \bar{u}, \bar{u}')_{\mathbb{R}^n}, \quad (56)$$

$$(\mu \mathbf{T}, \nabla u')_{L_2(\Omega)^3} = (D \bar{T}, \bar{u}')_{\mathbb{R}^n}, \quad (57)$$

$$(\sigma^{-1} \nabla \times \mathbf{T}, \nabla \times \mathbf{T}')_{L_2(\Omega)^3} = (C \bar{T}, \bar{T}')_{\mathbb{R}^m}, \quad (58)$$

$$(\mu \mathbf{T}, \mathbf{T}')_{L_2(\Omega)^3} = (M \bar{T}, \bar{T}')_{\mathbb{R}^m} \quad (59)$$

where  $n$  and  $m$  represent the total number of ‘active’ (non-Dirichlet) degrees of freedom for the scalar potential  $u$  and the vector potential  $\mathbf{T}$ , respectively.

#### 4.4. Matrix properties of the FE system for the 2.5D problem

**Theorem 8.** Matrix blocks  $N, C$  and  $M$  are strictly positive definite.

*Proof.* Nonnegative definiteness of  $N$  and  $C$ , as well as positive definiteness of  $M$ , follow directly from their definitions (56), (58), (59). Strict positive definiteness is due to the fact that for nonzero functions sinusoidal in the angular direction as in (40)–(42) the angular derivative cannot vanish:

$$\begin{aligned} (\nabla u)_\phi &= \frac{k}{r} u, \\ (\nabla \times \mathbf{T})_r &= \frac{k}{r} T_z; \quad (\nabla \times \mathbf{T})_z = -\frac{k}{r} T_r. \end{aligned} \quad \square$$

<sup>7</sup> In practice, first or higher order nodal elements are used for the approximation of  $u$  and either node or edge elements are used for  $\mathbf{T}$ .

**Theorem 9.** The block matrix

$$Z = \begin{pmatrix} N & -D \\ -D^T & M \end{pmatrix} \tag{60}$$

is strictly positive definite.

*Proof.* Due to (56)–(60), we have

$$\begin{aligned} \left( Z \begin{pmatrix} \bar{u} \\ \bar{T} \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{T} \end{pmatrix} \right)_{E^m} &= (N\bar{u}, \bar{u})_{E^n} - (D\bar{T}, \bar{u})_{E^n} - (D^T\bar{u}, \bar{T})_{E^m} + (M\bar{T}, \bar{T})_{E^m} \\ &= (\mu\nabla u, \nabla u) - (\mu\mathbf{T}, \nabla u) - (\mu\nabla u, \mathbf{T}) + (\mu\mathbf{T}, \mathbf{T}) \\ &= (\mu(\mathbf{T} - \nabla u), \mathbf{T} - \nabla u) > 0 \quad \text{for } \{u, \mathbf{T}\} \neq 0 \end{aligned} \tag{61}$$

where all unlabeled inner products are those of  $L_2(\Omega)^3$ . The last inequality is strict because  $\mathbf{T}$ , by construction (48), does not have a  $\varphi$ -component, whereas  $\nabla u$  does (43).  $\square$

4.5. *Stability properties of the DA system for the 2.5D problem*

System (55) has the form of (17) with

$$U = \begin{pmatrix} 0 & 0 \\ -D^T & M \end{pmatrix}, \tag{62}$$

$$V = \begin{pmatrix} N & -D \\ 0 & C \end{pmatrix}. \tag{63}$$

Since both  $N$  and  $C$  are positive definite,  $V$  is nonsingular.

**Theorem 10.** For the 2.5D problem, matrix  $V^{-1}U$ , with  $U$  and  $V$  given by (62), (63), is nondefective.

*Proof.* The respective eigenvalue problem is

$$\begin{vmatrix} -\lambda N & \lambda D \\ -D^T & M - \lambda C \end{vmatrix} = 0. \tag{64}$$

In the determinant above,  $(-\lambda)^n$  can be factored out and by theorem 9 the remaining determinant does not have  $\lambda = 0$  as its root.

Thus the multiplicity of  $\lambda = 0$  is exactly  $n$ . But  $n$  null vectors of  $U$  are easily determined: each of them corresponds to the potential  $u$  that is nonzero at one of the FE nodes and zero elsewhere.  $\square$

**5. Conclusion**

Two types of eddy current problems have been considered: the 2D coupled field-circuit problem and the ‘2.5D’ problem. Properties of the differential–algebraic systems

arising from FE discretization of these problems have been analyzed, and the main results can be summarized as follows.

For the 2D problem:

- The governing matrix of the full  $3 \times 3$  block differential–algebraic system for the one-component magnetic vector potential, the scalar potential gradient and the circuit mesh currents is not defective, provided the circuits are not degenerate.
- However, for the special case of the truncated system with given total currents in massive conductors, the DA system has index one; the structure of Jordan blocks corresponding to zero eigenvalues has been specified completely.
- The DA system can be made purely differential by switching to the time-integrated electric scalar potential. This formulation should be more widely adopted by researchers in engineering and applied electromagnetism.

For the 2.5D problem:

- Matrix properties of the DA system of the magnetic scalar potential method, with the two-component vector potential of eddy currents, have been analyzed.
- In contrast with the  $2 \times 2$  block system for the 2D problem with given total currents, the governing matrix for the 2.5D problem is not defective. The underlying reason is that the vector potential is effectively ‘gauged’ by setting its angular component to zero; this makes one of the key matrices strictly positive definite rather than nonnegative definite.

Various other formulations of time-dependent eddy current problems in two and three dimensions can be studied along the same lines.

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